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Planar Three-Body Problem in Rendezvous Coordinates

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Introduction

THE general three-body problem has a long history[1–4] with contributions from many researchers. It is well known that the general problem cannot be solved analytically.[5] However some special cases, such as the restricted three-body problem (where one of the three bodies has negligible mass as compared to the other two) or the general problem with two equal masses, have some particular solutions.[6–9] Under these circumstances it is appropriate to reformulate the general problem in special coordinate systems so that proper applications[10–12] can be solved accurately either numerically[13] or via perturbations.

An important application of the three-body problem is the rendezvous question[14–16] between a celestial body and a spacecraft in the presence of another large central body (e.g., a rendezvous between a large asteroid and spacecraft far from any planet in the gravitational field of the sun). Such a problem belongs to the class of the restricted three-body problems. In a previous paper[9] we reformulated this problem in "rendezvous coordinates," that is, in terms of the relative position of the spacecraft with respect to the celestial body it attempts to intercept. We were able also to reduce this problem to a simple system of differential equations by some transformations.

In recent years, however, there was a growing concern about the possible collision between Earth and a "large" asteroid in near-Earth orbit. Such a collision might cause a catastrophe to the biosphere on Earth. This (and other similar problems) motivates us to consider in this Note the motion of two celestial bodies of similar masses in the presence of a third large body (e.g., the sun) and rewrite the equations of motion in terms of the relative position between these two celestial bodies. Such a reformulation of the problem can determine whether the two bodies are on a collision (or capture) course and the minimum distance that will be attained between them. Thus in this Note we are treating the rendezvous problem within the context of the general two-dimensional three-body problem rather than just the restricted three-body problem, which was considered in Ref. 9.

From a numerical point of view, as the distance between the two colliding bodies decreases the gravitational attraction between them increases, and this might lead to inaccuracies in the numerical solution of this problem. This "regularization" problem was addressed in the past by at least two authors.[10,11] In both cases complex coordinates were introduced. The problem becomes then somewhat physically intractable, and only the combined use of symbolic and numerical algorithms allows the final derivation of the equations and their simulation.[13] As a result, some physical insight into this problem is lost. The equations derived in this Note do not address the regularization problem. However, the equations are formulated in terms of physical coordinates, and the independent variable is one of the angle variables. As a result, the problem is reduced to seven rather than eight (first-order) equations, and one of the dependent variables is the angular momentum of one of the bodies in the system. Furthermore, the long-term integration of these equations remains stable.

The plan of this Note is as follows. At first we present the general formulation of the problem and derive the orbit equations for the colliding bodies. We then recast the equations of motion in rendezvous coordinates and apply several transformations to simplify them. Two special cases of the general problem are considered next. The first is the restricted three-body problem. The second considers the case where two of the three bodies in the system have equal mass. Finally we discuss the numerical solution of the rendezvous equations and end up with summary and conclusions.

Formulation of the Problem

In this Note we consider the three-body problem under the assumption that the mass of the central body (the primary, whose mass is M) is much larger than the masses of the other two bodies in the system whose mass are m_1 and m_2, respectively, and $m_1 < m_2$. (However, we do not assume that m_1 is insignificant as compared to m_2.) With these assumptions the (approximate) equations of motion of m_1, m_2 are

$$\ddot{\boldsymbol{r}}_1 = -\left(GM/r_1^3\right)\boldsymbol{r}_1 - \left(Gm_2/r_{12}^3\right)\boldsymbol{r}_{12}$$
$$r_1 = |\boldsymbol{r}_1|, \qquad r_{12} = |\boldsymbol{r}_{12}| \tag{1}$$

$$\ddot{\boldsymbol{r}}_2 = -\left(GM/r_2^3\right)\boldsymbol{r}_2 + \left(Gm_1/r_{12}^3\right)\boldsymbol{r}_{12}, \qquad r_2 = |\boldsymbol{r}_2| \tag{2}$$

In these equations $\boldsymbol{r}_1, \boldsymbol{r}_2$ are the positions of m_1, m_2 with respect to the primary and $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

Taking the vector products of Egs. (1) and (2) with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively, we obtain

$$\boldsymbol{r}_1 \times \ddot{\boldsymbol{r}}_1 = \left(Gm_2/r_{12}^3\right)\boldsymbol{r}_1 \times \boldsymbol{r}_2 \tag{3}$$

$$\boldsymbol{r}_2 \times \ddot{\boldsymbol{r}}_2 = \left(Gm_1/r_{12}^3\right)\boldsymbol{r}_2 \times \boldsymbol{r}_1 \tag{4}$$

Hence

$$\dot{\boldsymbol{L}}_1 = \left(Gm_2/r_{12}^3\right)\boldsymbol{r}_1 \times \boldsymbol{r}_2 \tag{5}$$

$$\dot{\boldsymbol{L}}_2 = -\left(Gm_1/r_{12}^3\right)\boldsymbol{r}_1 \times \boldsymbol{r}_2 \tag{6}$$

where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are the angular momentum of m_1 and m_2, respectively. From these equations we infer that if the initial $\boldsymbol{L}_1, \boldsymbol{L}_2$ are orthogonal to the plane of $\boldsymbol{r}_1, \boldsymbol{r}_2$, then they will remain always orthogonal to this plane. It follows then that $\boldsymbol{L}_1, \boldsymbol{L}_2$ have constant direction and the motion remains planar. In the following we consider only the planar three-body problem and assume that $\boldsymbol{L}_1, \boldsymbol{L}_2$ are along the z axis). We observe, however, that neither $\boldsymbol{L}_1$ nor $\boldsymbol{L}_2$ is conserved in this problem. However, if $m_1 = m_2$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_1 + \boldsymbol{L}_2) = 0 \tag{7}$$

i.e., $\boldsymbol{L}_1 + \boldsymbol{L}_2 = \text{constant}$.

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If we now let the polar coordinates of m_1 and m_2 be (r_1, ϕ) and (r_2, θ), respectively, we then have from Eq. (6) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(r_2^2\dot\theta\right) = -\frac{Gm_1}{r_{12}^3}(\boldsymbol{r}_1\times\boldsymbol{r}_2)\cdot\boldsymbol{k} \tag{8}$$

Integrating this equation with respect to time yields

$$r_2^2\dot\theta = -\int\frac{Gm_1}{r_{12}^3}(\boldsymbol{r}_1\times\boldsymbol{r}_2)\cdot\boldsymbol{k}\,\mathrm{d}t = A \tag{9}$$

where $A = A(r_1, r_2, \theta, \phi)$. However if we assume that the relation $\theta=\theta(t)$ can be inverted, at least locally, then formally $A = A(\theta)$, and we can rewrite the relation (9) in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{A(\theta)}{r_2^2} \tag{10}$$

Using this relation, we can rewrite the radial equation of motion for m_2

$$\ddot r_2 - r_2\dot\theta^2 = -GM\big/r_2^2 + \left(Gm_1\big/r_{12}^3\right)\boldsymbol{r}_{12}\cdot\boldsymbol{r}_2/r_2 \tag{11}$$

in the form

$$r_2\left(A^2r_2'\right)' - 2A^2(r_2')^2 = A^2r_2^2 - GMr_2^3 + \left(Gm_1\big/r_{12}^3\right)(\boldsymbol{r}_{12}\cdot\boldsymbol{r}_2)r_2^4 \tag{12}$$

where primes denote differentiation with respect to θ. This is the orbit equation for m_2. A similar equation can be derived for m_1.

Equation for $\boldsymbol{r}_{12}$

In this section we derive the equations of motion in rendezvous coordinates, that is, replace the equation for $\boldsymbol{r}_1$ by an appropriate (but approximate) equation for $\boldsymbol{r}_{12}$. Rewriting $\boldsymbol{r}_1 = \boldsymbol{r}_2 + \boldsymbol{r}_{12}$ and substituting in Eq. (1), we obtain

$$\ddot{\boldsymbol{r}}_2 + \ddot{\boldsymbol{r}}_{12} = -\left[GM\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3\right](\boldsymbol{r}_2+\boldsymbol{r}_{12}) - \left(Gm_2\big/r_{12}^3\right)\boldsymbol{r}_{12} \tag{13}$$

Using Eq. (2), this yields

$$\ddot{\boldsymbol{r}}_{12} = -GM\left[1\big/(\boldsymbol{r}_1+\boldsymbol{r}_{12})^3 - 1\big/r_2^3\right]\boldsymbol{r}_2 - \left[GM\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3 + G(m_1+m_2)\big/r_{12}^3\right]\boldsymbol{r}_{12} \tag{14}$$

This is the equation for $\boldsymbol{r}_{12}$ in the inertial coordinate system. We now transform this equation to one that is centered in m_2. This leads to

$$\ddot{\boldsymbol{r}}_{12} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}_{12} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_{12}) + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\boldsymbol{r}_{12} = -GM\left[\frac{1}{(\boldsymbol{r}_1+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]\boldsymbol{r}_2 - \left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]\boldsymbol{r}_{12} \tag{15}$$

where $\Omega = (0, 0, \omega)$. We now introduce a coordinate system attached to m_2 so that the x_1 axis is tangential but opposed to its motion. The x_2 axis is in the direction of $\boldsymbol{r}_2$, and x_3 completes a right-handed system.(This coordinate system was used in Refs. 16 and 17 to discuss the rendezvous problem of a spacecraft with a satellite in the gravitational field of the Earth.) In this frame $\boldsymbol{r}_{12} = (x, y, 0)$ and $\boldsymbol{r}_2 = r_2(0, 1, 0)$. Rewriting Eq. (15) in this frame leads to the following system of equations:

$$\ddot x - 2w\dot y - \omega^2x - \dot\omega y = -\left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]x \tag{16}$$

$$\ddot y + 2\omega\dot x - \omega^2y + \dot\omega x = -GM\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]r_2 - \left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]y \tag{17}$$

Transformation of the Rendezvous Equations

To reduce Eqs. (16) and (17), we first make a change of the independent variable from time to θ. This leads to

$$\omega x'' + \omega'x' - \omega x - 2\omega y' - \omega' y = -\frac{1}{\omega}\left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]x \tag{18}$$

$$\omega y'' + \omega'y' - \omega y + 2\omega x' + \omega' x = -\frac{GM}{\omega}\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]r_2 - \frac{1}{\omega}\left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]y \tag{19}$$

where primes denote differentiation with respect to θ. These can be rewritten as

$$(\omega x')' - 2\omega^{\frac12}(\omega^{\frac12}y)' - \omega x = -\frac{1}{\omega}\left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]x \tag{20}$$

$$(\omega y')' + 2\omega^{\frac12}\left(\omega^{\frac12}x\right)' - \omega y = -\frac{GM}{\omega}\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]r_2 - \frac{1}{\omega}\left[\frac{GM}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} + \frac{G(m_1+m_2)}{r_{12}^3}\right]y \tag{21}$$

We now introduce

$$u = \omega^{\frac12}x, \qquad v = \omega^{\frac12}y \tag{22}$$

After some long algebra, Eqs. (20) and (21) take the form

$$u'' - 2v' = \left[1 + \tfrac12(\omega''/\omega) - \tfrac14(\omega'/\omega)^2 - GM\big/r_2^3\omega^2\right]u - (1/\omega)\left\{GM\left[1\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3 - 1\big/r_2^3\right] + G(m_1+m_2)\big/r_{12}^3\right\}u \tag{23}$$

$$v'' + 2u' = \left[1 + \tfrac12(\omega''/\omega) - \tfrac14(\omega'/\omega)^2 - GM\big/r_2^3\omega^2\right]v + -\left(GM\big/\omega^{\frac32}\right)\left[1\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3 - 1\big/r_2^3\right]r_2 - (1/\omega^2)\left\{GM\left[1\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3 - 1\big/r_2^3\right] + G(m_1+m_2)\big/r_{12}^3\right\}v \tag{24}$$

where

$$\boldsymbol{\sigma} = (u, v) = \omega^{\frac12}\boldsymbol{r}_{12} \tag{25}$$

Using Eq. (10) to substitute for ω and its derivatives, we have

$$1 + \tfrac12(\omega''/\omega) - \tfrac14(\omega'/\omega)^2 - GM\big/r_2^3\omega^2 = 1 + \left[\tfrac12(A''/A) - \tfrac14(A'/A)^2 - A'r_2'/Ar_2 - r_2''/r_2 + 2(r_2'/r_2)^2\right] - (GM/A^2)r_2 \tag{26}$$

However, from the orbit equation and Eq. (10), we have also

$$1 - \frac{r_2''}{r_2} - \frac{A'r_2'}{Ar_2} + 2\left(\frac{r_2'}{r_2}\right)^2 - \frac{GMr_2}{A^2} = -\frac{Gm_1}{r_{12}^3}\frac{(\boldsymbol{r}_{12}\cdot\boldsymbol{r}_2)}{r_2^2\omega^2} \tag{27}$$

$$u'' - 2v' = \left[\frac12\frac{A''}{A} - \frac14\left(\frac{A'}{A}\right)^2 - \frac{Gm_1}{A\sigma^3}(\boldsymbol{\sigma}\cdot\boldsymbol{r}_2)\right]u - \frac{G(m_1+m_2)}{\omega^{\frac12}\sigma^3}u - \frac{GM}{\omega^2}\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]u \tag{28}$$

$$v'' + 2u' = \left[\frac{1}{2}\frac{A''}{A} - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{Gm_1}{A\sigma^3}(\boldsymbol{\sigma}\cdot\boldsymbol{r}_2)\right]v$$
$$-\frac{GM}{\omega^{\frac{3}{2}}}\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]r_2$$
$$-\frac{G(m_1+m_2)}{\omega^{\frac{1}{2}}\sigma^3}v - \frac{1}{\omega^2}GM\left[\frac{1}{(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3} - \frac{1}{r_2^3}\right]v \tag{29}$$

We can simplify these equations if we assume that $r_{12} \ll r_1$, $r_{12} \ll r_2$ and use a first-order Taylor expansion to make the following approximation:

$$1\big/(\boldsymbol{r}_2+\boldsymbol{r}_{12})^3 \cong \left(1\big/r_2^3\right)\left\{1 - 3\left[(\boldsymbol{r}_2\cdot\boldsymbol{r}_{12})\big/r_2^2\right]\right\} \tag{30}$$

Substituting this approximation in Eqs. (28) and (29), we finally obtain

$$u'' - 2v' = \left[\frac{1}{2}\frac{A''}{A} - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{Gm_1}{A\sigma^3}(\boldsymbol{\sigma}\cdot\boldsymbol{r}_2)\right]u - \frac{G(m_1+m_2)}{\omega^{\frac{1}{2}}\sigma^3}u \tag{31}$$

$$v'' + 2u' = \left[\frac{1}{2}\frac{A''}{A} - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{Gm_1}{A\sigma^3}(\boldsymbol{\sigma}\cdot\boldsymbol{r}_2)\right]v$$
$$+\frac{3GM}{r_2^3\omega^2}v - \frac{G(m_1+m_2)}{\omega^{\frac{1}{2}}\sigma^3}v \tag{32}$$

These equations together with Eqs. (10) and (12) provide the required formulation of our problem in rendezvous coordinates. (For further elaboration of the numerical solution of this system, the Numerical Formulation section.)

Two Special Cases

Restricted Three-Body Problem

When m_1 has negligible mass, A can be treated as a constant, and the terms containing m_1 in Eqs. (28) and (29) can be neglected. Under these conditions Eqs. (28) and (29) form an independent system for the motion of m_1 provided that the orbit of m_2 is known. Such a situation arises for example when one considers the motion of a satellite orbiting in the Earth–moon system or a spacecraft from Earth traveling toward the moon. Further discussion of the resulting equations was carried out by the present author in a previous publication.[9]

$m_1 = m_2$

When $m_1 = m_2$, it is advantageous to reformulate the problem in term of $\boldsymbol{R} = \boldsymbol{r}_1 + \boldsymbol{r}_2$ and $\boldsymbol{r} = \boldsymbol{r}_{12}$. Taking the sum and difference of Eqs. (1) and (2), we obtain

$$\ddot{\boldsymbol{R}} = -GM\left\{\boldsymbol{r}_1\big/r_1^3 + \boldsymbol{r}_2\big/r_2^3\right\} \tag{33}$$

$$\ddot{\boldsymbol{r}} = -(2Gm/r^3)\boldsymbol{r} - GM\left\{\boldsymbol{r}_1\big/r_1^3 - \boldsymbol{r}_2\big/r_2^3\right\} \tag{34}$$

Assuming that $r \ll R$, we can apply the following approximations to these equations:

$$1\big/r_1^3 = 8/(\boldsymbol{R}+\boldsymbol{r})^3 \cong (8/R^3)\left\{1 - \tfrac{3}{2}[(r/R)^2 + (2\boldsymbol{r}\cdot\boldsymbol{R})/R^2]\right\} \tag{35}$$

$$1\big/r_2^3 = 8/(\boldsymbol{R}-\boldsymbol{r})^3 \cong (8/R^3)\left\{1 - \tfrac{3}{2}[(r/R)^2 - (2\boldsymbol{r}\cdot\boldsymbol{R})/R^2]\right\} \tag{36}$$

This leads to

$$\ddot{\boldsymbol{R}} = -(8GM/\boldsymbol{R}^3)\left\{\boldsymbol{R} - \tfrac{3}{2}(r/R)^2\boldsymbol{R} - [3(\boldsymbol{r}\cdot\boldsymbol{R})/R^2]\boldsymbol{r}\right\} \tag{37}$$

$$\ddot{\boldsymbol{r}} = -(2Gm/r^3)\boldsymbol{r} - 8GM\left\{\boldsymbol{r} - \tfrac{3}{2}(r/R)^2\boldsymbol{r} - 3[(\boldsymbol{r}\cdot\boldsymbol{R})/R^2]\boldsymbol{R}\right\} \tag{38}$$

Neglecting the nonlinear terms in $\boldsymbol{r}$, we finally obtain

$$\ddot{\boldsymbol{R}} = -(8GM/R^3)\boldsymbol{R} \tag{39}$$

$$\ddot{\boldsymbol{r}} = -(2Gm/r^3)\boldsymbol{r} - (8GM/R^3)\left\{\boldsymbol{r} - [3(\boldsymbol{r}\cdot\boldsymbol{R})/R^2]\boldsymbol{R}\right\} \tag{40}$$

We deduce then that under the approximations introduced here $\boldsymbol{R}$ traces out a Keplerian orbit. In a coordinate system rotating with $\boldsymbol{R}$ (similar to the centered at m_2, which was introduced in the third section), Eq. (40) becomes

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\boldsymbol{r} = -\frac{2Gm}{r^3}\boldsymbol{r} - \frac{8GM}{R^3}\left[\boldsymbol{r} - \frac{3(\boldsymbol{r}\cdot\boldsymbol{R})}{R^2}\boldsymbol{R}\right] \tag{41}$$

The reduction and solution of this equation has been treated at length in Ref. 14. (Similar treatment of this equation in a general central force was carried out in Ref. 17.)

Further elaboration of the two special cases discussed in this section will be carried out in a separate publication.

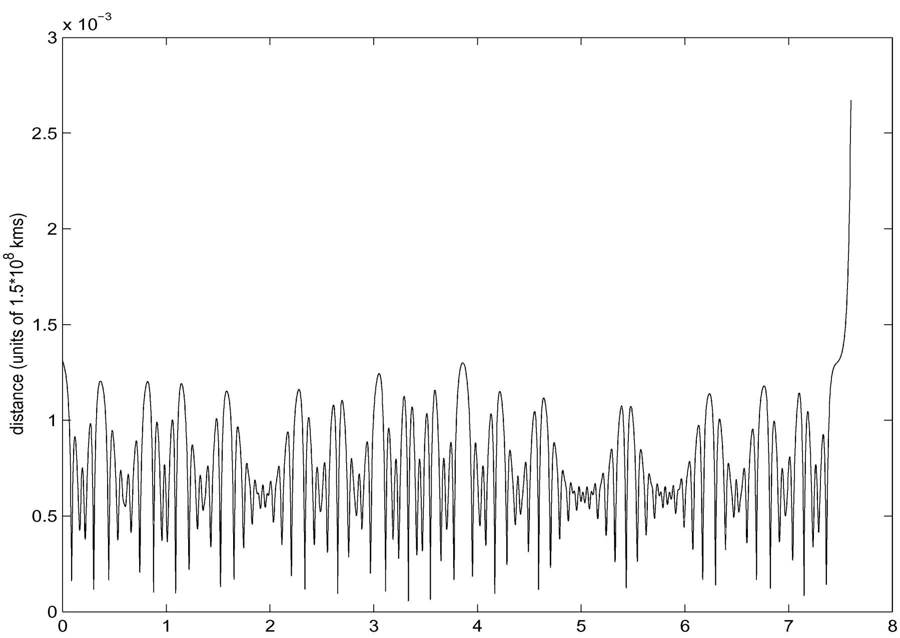

Fig. 1 Orbit of m_1 around m_2 until it is released because of its large angular momentum and by a proper alignment of the three bodies.

Numerical Formulation

To carry out numerical simulations of the three-body problem in rendezvous coordinates, we combine Eqs. (8) and (9) and the definition of $\boldsymbol{r}_{12}$ to obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{Gm_1}{r_{12}^3}\boldsymbol{r}_{12} \times \boldsymbol{r}_2 \cdot \boldsymbol{k} \tag{42}$$

Using a polar representation of $\boldsymbol{r}_2$ and expressing $\boldsymbol{r}_{12}$ in terms of $\boldsymbol{\sigma}$, this leads to

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{Gm_1\omega}{\sigma^3} r_2 u \tag{43}$$

Changing the independent variable from t to θ, we finally obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -\frac{Gm_1 r_2}{\sigma^3} u \tag{44}$$

Equations (12), (28), (29) (with ω being replaced by A/r_2^2), and (44) form a complete system for the numerical solution of the three-body problem in terms of seven first-order equations. However if $r_{12} \ll r_1$, $r_{12} \ll r_2$, we can replace Eqs. (28) and (29) by Eqs. (30) and (31).

As an actual application of these equations, we consider the sun–Earth–moon system.[18]

As is well known, there are several competing theories regarding the origin of the Earth–moon system. One of this theories is that the moon was captured by Earth in the early evolution of the solar system. Such a theory explains naturally the composition difference between the Earth and the moon (which does not have an iron core). However, there are some major objections to this theory, foremost of which is the slowdown in the moon velocity that is needed for this capture to happen.

In the following we simulate the three-body problem for a system in which m_1 and m_2 have the same masses as the moon and Earth using the equations derived in this Note. To initiate these

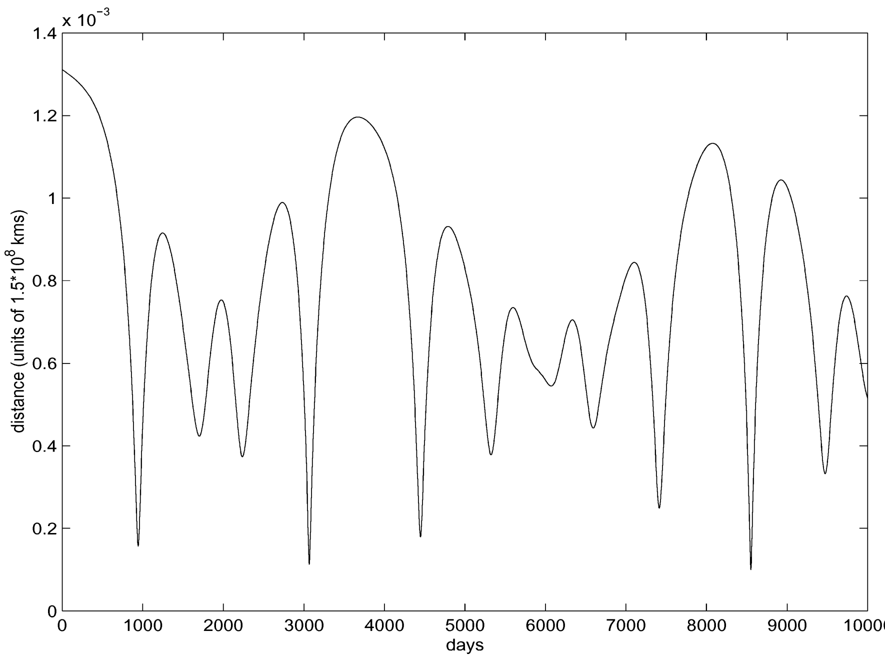

Fig. 2 Orbit of m_1 around m_2 for the first 10^4 days of its capture (second simulation).

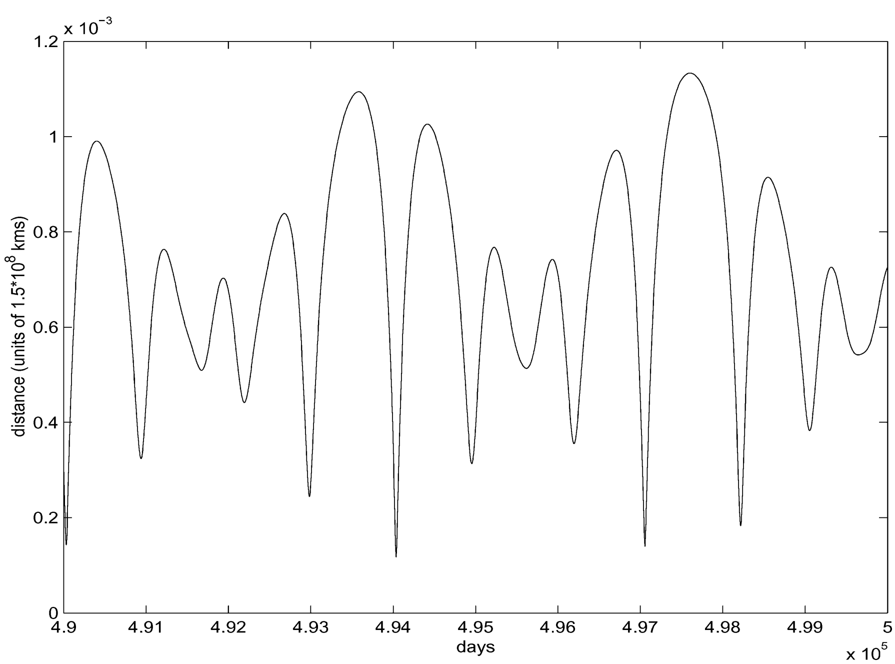

Fig. 3 Orbit of m_1 around m_2 for the last 10^4 days of the second simulation.

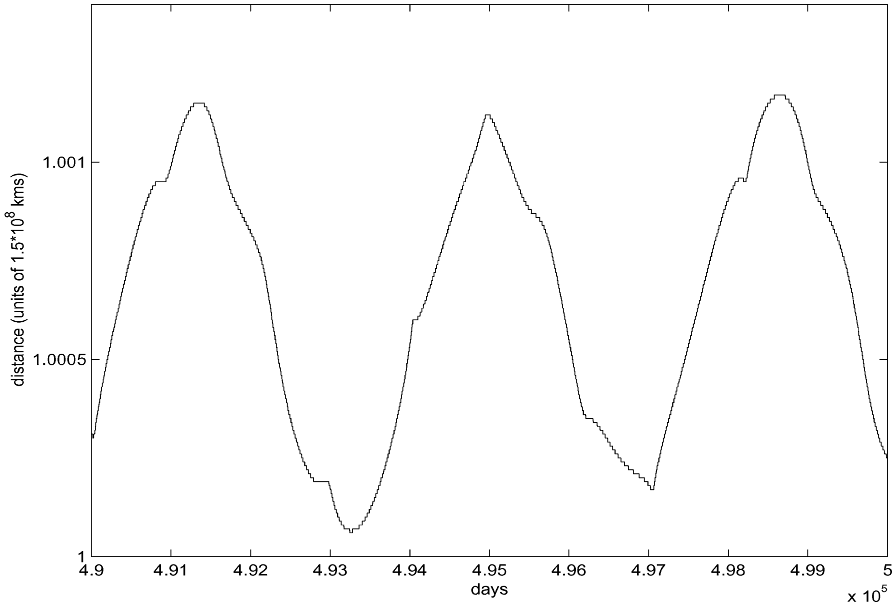

Fig. 4 Impact of the m_1 capture on the orbit of m_2 during the last 10^4 days of the second simulation.

simulations, we assume that a time zero m_2 (Earth) is in a stable circular orbit at a distance of 1.5×10^8 km from the sun. We then used various initial conditions for the relative distance and velocity of m_1 to determine those that will lead to the capture of m_1 by m_2 and the impact of this capture on the orbit of m_2. The time step in these simulations was one day (86,400 s, which translates to $\Delta\theta$ of 0.00172) for a duration 5×10^5 days. (In the simulations we used the initial radius of the Earth orbit as a unit length.) To integrate the equations, we used the Runge–Kutta algorithm of order 7–8 with tolerance of 10^{-8} at each step. The results of two of these simulations are presented in Figs. 1–4. In both of these simulations, the initial relative distance was 1.5×10^6 km, and the initial relative velocity (of the moon with respect to the Earth) was (0, 1.3), (0, 1) km/s, respectively.

For the first set of initial conditions, we obtained a temporary capture, that is, the moon was captured for some time in Earth orbit, but eventually with proper alignment of the three bodies it was released from Earth orbit. (Figure 1 shows the orbit of m_1 until it is released.) In the second simulation m_1 is captured permanently. Figures 2 and 3 show its orbit for the first and last 10^4 days of the simulation. Figure 4 shows the impact of of this capture on the orbit of m_2.

From these simulations we infer that the initial relative velocity of the moon with respect to the Earth must be close to 1 km/s for such a capture to take place.

Conclusions

The three-body problem is a classical problem that has attracted a lot of attention in the past and is the subject of ongoing research. In this Note we concentrated on the practical aspects of this problem in view of recent "near-miss collisions" between Earth and asteroids in near-Earth orbit. In these situations it is important to solve the problem in rendezvous coordinates that are robust from a numerical and practical point of view. In particular our formulation will be useful when, initially, the two bodies are not too close to each other, and one would like to compute the long-term evolution of the system.

From a theoretical point of view, the equations we derived for the general three-body problem show clearly how the corresponding equations for the restricted three-body problem are obtained as a limit of the more general equations.

References

[1]Poincare, H., "Sur les Solutions Periodiques et le Principe de Moindre Action," *Comptes Rendus de l'Academic des Sciences*, Vol. 123, 1896, pp. 915–918.

[2]Lagrange, J. L., "Essai sur le Probleme des Trois Corps," *Oeuvres VI*, 1772, pp. 229–324.

[3]Siegel, C. L., and Moser, J. K., *Lectures on Celestial Mechanics*, Springer-Verlag, New York, 1971.

[4]Marchal, Ch., *The Three Body Problem*, Elsevier, Amsterdam, 1990.

[5]Tsygvintsev, A., "The Meromorphic Non-Integrability of the Planar Three-Body Problem," *Comptes Rendus de l'Academic des Sciences*, *Serie 1: Mathematique*, Vol. 331, No. 3, 2000, pp. 241–244.

[6]Montgomery, R., "A New Solution to the Three-Body Problem," *Notices of the American Mathematical Society*, Vol. 48, No. 5, 2001, pp. 471–481.

[7]Montgomery, R., and Chenciner, A., "A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses," *Annals of Mathematics*, Vol. 152, No. 3, 2000, pp. 881–901.

[8]Quesne, C., "An Exactly Solvable Three-Particle Problem with Three-Body Interaction," *Physical Review A*, Vol. 55, No. 5, 1997, pp. 3931–3934.

[9]Humi, M., "Approximate Equations for the Coplanar Restricted Three Body Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 1, 2002, pp. 182–184.

[10]Levi-Civita, T., "Sur la Regularisation du Probleme de Trois Corps," *Acta Mathematica*, Vol. 42, No. 1, 1920, pp. 99–144.

[11]J. Waldvogel, J., "A New Regularization of the Planar Problem of Three Bodies," *Celestial Mechanics*, Vol. 6, No. 2, 1972, pp. 221–231.

[12]Malige, F., Robutel, P., and Lasker, J., "Partial Reduction in the N-Body Problem Using the Angular Momentum Integral," *Celestial Mechanics*, Vol. 84, No. 3, 2002, pp. 283–316.

[13]Waldvogel, J., "Orbits in the Planar Three Body Problem," *Solving Problems in Scientific Computing Using Maple and MATLAB*, 3rd edition, edited by W. Gander and J. Hrebicek, Springer-Verlag, New York, 2002, pp. 50–79.

[14]Carter, T., and Humi, M., "Fuel Optimal Rendezvous near a Point in General Keplerian Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 567–573.

[15]Humi, M., and Carter, T., "Fuel Optimal Rendezvous in a Central-Force Field with Linear Drag," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 1, 2002, pp. 74–79.

[16]Carter, T., and Humi, M., "Clohessy-Wiltshire Modified to Include Quadratic Drag," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 6, 2002, pp. 1058–1063.

[17]Humi, M., "Fuel Optimal Rendezvous in a General Central Force Field," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 1, 1993, pp. 215–217.

[18]Prussing, J. E., and Conway, B. A., *Orbital Mechanics*, Oxford Univ. Press, Oxford, 1993.